MATH2048 Honours Linear Algebra II

Midterm Examination 2

Please show all your steps, unless otherwise stated. Answer all **five** questions.

1. Let $V = M_{2\times 2}(\mathbb{R})$ and $T \in \mathcal{L}(V)$ is defined by
$$T(A) = A + A^T$$
Determine whether T is diagonalizable. Please explain your answers with details.

2. Let $V = P_2(\mathbb{R})$ and $T \in \mathcal{L}(V)$ is defined by
$$T(a + bx + cx^2) = (-a - 2b + c) - (\frac{1}{2}c)x + (2b + 2c)x^2.$$
   (a) Find a polynomial $g(t)$ of degree at most 2 such that $T^3 = g(T)$. (**Hint:** Cayley-Hamilton Theorem.)

   (b) Let $\mathbf{v} = -x + 2x^2 \in V$ and W be the T-cyclic subspace of V generated by $\mathbf{v}$. Show that $T^2(\mathbf{v}) = a_0\mathbf{v} + a_1 T(\mathbf{v})$ for some $a_0,\ a_1 \in \mathbb{R}$. What's $\dim(W)$? Find the characteristic polynomial of $T|_W$, the restriction of T to W.

3. Let V be a vector space over $\mathbb{C}$ with an ordered basis $\beta = \{\mathbf{v}_0, \mathbf{v}_1, ..., \mathbf{v}_{n-1}\}$. Define a linear operator $T : V \to V$ by:
$$T(\mathbf{v}_0) = \mathbf{v}_0 - \mathbf{v}_{n-1} \text{ and } T(\mathbf{v}_k) = \mathbf{v}_k - \mathbf{v}_{k-1} \text{ for } 1 \le k \le n-1.$$
Let $\omega_k = e^{i\frac{2\pi k}{n}} = \cos(\frac{2\pi k}{n}) + i\sin(\frac{2\pi k}{n})$ for any integer k (where $i = \sqrt{-1}$).

   (a) Show that $\mathbf{u}_k = \sum_{j=0}^{n-1} \omega_k^j \mathbf{v}_j$ is an eigenvector of T for any integer k and show that T is diagonalizable. (**Hint:** You may use the fact that $\omega_k^j = e^{i\frac{2\pi kj}{n}}$ and $\omega_k^n = 1$.)

   (b) Now, consider the linear operator $U : V \to V$ defined by: $U(\mathbf{v}_0) = \mathbf{v}_1 - 2\mathbf{v}_0 + \mathbf{v}_{n-1}$, $U(\mathbf{v}_k) = \mathbf{v}_{k-1} - 2\mathbf{v}_k + \mathbf{v}_{k+1}$ for $1 \le k \le n-2$ and $U(\mathbf{v}_{n-1}) = \mathbf{v}_{n-2} - 2\mathbf{v}_{n-1} + \mathbf{v}_0$. Using (a), determine if there exists an ordered basis γ for V such that $[U]_\gamma$ is a **real** diagonal matrix. Please explain you answer with details.

4. Let F be a field and $V = F^n$ be a vector space over F. Let $\Phi : V^* \to F^n$ be defined by $\Phi(f) = (c_1, \ldots, c_n)$, where $f(\vec{x}) = f(x_1, \ldots, x_n) = \sum_{i=1}^n c_i x_i$. Let $W = \{\vec{x} = (x_1, \ldots, x_n) \in V : \sum_{i=1}^n x_i = 0\}$ be a subspace of V.

   (a) Let $\vec{v}_0 = (1, 1, \ldots, 1) \in V$. Let $\eta : W^* \to V^*$ be a linear map between W^* and V^* such that $\eta(g)(\vec{x}) = \begin{cases} g(\vec{x}) & \vec{x} \in W \\ 0 & \vec{x} \in \text{span}(\{\vec{v}_0\}) \end{cases}$. Show that η is well-defined. (That is, for each $g \in W^*$, $\eta(g) \in V^*$ is uniquely determined.)

   (b) Show that $R(\Phi \circ \eta) = \{(c_1, \ldots, c_n) \in F^n : \sum_{i=1}^n c_i = 0\}$.

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5. Let V be a finite-dimensional vector space over $\mathbb{R}$ with an ordered basis $\beta = \{\mathbf{v}_i\}_{i=1}^n$. Consider a linear transformation $\Phi : V \otimes V \to \mathcal{L}(V^*, V)$, which is defined by:

$$\Phi(\mathbf{v}_i \otimes \mathbf{v}_j)(f) = f(\mathbf{v}_i)\mathbf{v}_j \text{ for all } f \in V^*.$$

   (a) Prove that Φ is an isomorphism and $[\Phi(\mathbf{w}_1 \otimes \mathbf{w}_2)]_{\beta^*}^{\beta} = ([\mathbf{w}_2]_\beta)([\mathbf{w}_1]_\beta)^T$.

   (b) Let $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$ be a linearly independent subset of V and A be a $m \times m$ real matrix. Consider $G = \sum_{i=1}^m \sum_{j=1}^m A_{ij}\mathbf{w}_i \otimes \mathbf{w}_j$, where A_{ij} is the i-th row j-th column entry of A. Find the rank of $\Phi(G)$.

END OF PAPER